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COMMENT

# Operator content of the multicritical magnetic hard-square model

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**Abstract.** The transfer matrix spectra of the magnetic hard squares along the multicritical line are analysed using the data for the strip width $N$ up to 12. The operator content of the model is exactly the same as that of the Ashkin-Teller model. When $N$ is odd, the spectrum is accounted for by the continuum theory in which the antiperiodic boundary condition is imposed to one of the Ising variables of the Ashkin-Teller model.

The magnetic hard-square (MHS) model is a lattice gas of Ising spins on the square lattice with nearest-neighbour exclusions. (Pearce 1985, 1987). The spin variable at each site takes the value 0 or $\pm 1$ according to whether the site is unoccupied or occupied with the corresponding sign of the spin. The Boltzmann factor of a face is given by

$$W(a, b, c, d) = \begin{cases} \exp\{L(a^2c^2 + b^2d^2) + K(ac + bd)\} z^{(a^2+b^2+c^2+d^2)/4} \\ \qquad ab = bc = cd = da = 0 \\ 0 \qquad \text{otherwise} \end{cases} \tag{1}$$

where $a$, $b$, $c$ and $d$ are the four spins on a face taken in anticlockwise order, $L(K)$ is the diagonal lattice-gas (spin) interaction and $z$ is the fugacity. Pearce (1985) obtained the free energy for the anisotropic version of the model in the thermodynamic limit on certain manifolds in the thermodynamic space. The so-called TII manifold for the isotropic model is a line of multicritical points parametrised by $\lambda$ for $0 \leqslant \lambda \leqslant 2\pi/3$ and is given by

$$\begin{aligned} &\tanh K = \sin(\tfrac{1}{2}\lambda)/\sin \lambda \\ &\exp(-L) = \cosh K \,(\tanh K)^2 \\ &z = (\tanh K)^4. \end{aligned} \tag{2}$$

Recently, Pearce and Kim (1987, hereafter referred to as I) found numerically that this multicritical line is characterised by the $c = 1$ central charge and is associated with the continuously varying exponents (Cardy 1987).

The thermal exponent $X_{\mathrm{T}}$ is given by

$$X_{\mathrm{T}} = 2/9(1 - \lambda/\pi). \tag{3}$$

The MHS model has the same symmetry as the Ashkin-Teller (AT) model and its first few scaling dimensions accurately determined in I are identical in form to those of the AT model. The partition function of a conformally invariant system on a finite torus is determined by the operator content of the transfer matrix (Cardy 1986). The full operator content of the AT model has since been obtained by Baake *et al* (1987), Yang (1987) and Saleur (1987).

In this comment, we extend the identification of the MHS transfer matrix spectra to higher levels and show that the operator content of the multicritical MHS model is exactly the same as that of the AT model. We also show that the MHS model on a strip with periodic boundary conditions but with an odd number of sites across the strip corresponds to the AT model in which the antiperiodic boundary condition is imposed to one of its spin variables. This is not surprising since lattice gases on a square lattice with strong nearest-neighbour repulsion can be mapped into antiferromagnetic Ising models in fields and an antiferromagnetic Ising model on a cylinder is the same as a ferromagnetic model with a seam of antiferromagnetic bonds when the number of spins around the cylinder is odd.

In I, the transfer matrix of the MHS model is diagonalised for the strip width $N$ up to 10 and the results are used to solve the inversion identity for higher $N$. In this work, we use the transfer matrix spectra for $N \leqslant 12$ for further identification of the levels. The normalised level $X_r(N)$ for even $N$ is defined to be

$$X_r(N) = (N/2\pi)\ln(\Lambda_0/|\Lambda_r|) \tag{4}$$

where the $\Lambda_r$ ($\leqslant \Lambda_0$, $r = 0, 1, 2, \ldots$) are the transfer matrix eigenvalues for a strip of width $N$ under the periodic boundary conditions. For odd $N$, the identity representation (see below) does not appear in the operator content. Accordingly, we define

$$X_r(N) = \frac{c}{12} + \frac{N}{2\pi}(-Nf - \ln|\Lambda_r|) \tag{5}$$

where $c = 1$, and $f$ is the bulk free energy per site. As $N \to \infty$, $X_r(N)$ approaches to the scaling dimension of the corresponding operator (Cardy 1986). Each level is classified by the spin-reversal quantum number $R$ ($=\pm1$), the sublattice interchange parity $R'(=\pm1)$, and the spin $S$. When $R' = -1$, the momentum $P$ and the spin $S$ are related by

$$P = \pi + (2\pi/N)S \tag{6}$$

(see I, Kim and Pearce 1987).

Table 1 shows identifications of several levels in the sectors $(R, R') = (1, 1)$, $(1, -1)$ and $(-1, 1)$ for $N$ even when $\lambda = 7\pi/12$ ($X_T = 8/15$). For even $N$, the $(-1, -1)$ sector is degenerate with the $(-1, 1)$ sector. This degeneracy originates from the two-dimensional irreducible representation of the group $D_4$. The first columns are the spins and the second columns are the values of $X_r$ (12). To each non-zero spin level, there is a degenerate level with the opposite sign of the spin. Only positive spins are displayed. The third columns are the extrapolated values of the sequences $X_r(N)$. Since our sequences are rather short, we fit the last three data to the form

$$X_r(N) = X + aN^{-b} \tag{7}$$

to obtain the extrapolated values shown in the table. This procedure is justified since the asymptotic form of $X_r(N)$ is expected to be of the form (7) (Cardy 1986, Reinicke 1987). For some levels, the data for $N = 8$ (7) for the even (odd) $N$ sequences do not reflect the asymptotic behaviour and give unphysical results. In such cases, we leave the third columns blank. The operator contents of the AT and the MHS models are described in terms of the irreducible representations $(\Delta, \bar{\Delta})$ of the two commuting $c = 1$ Virasoro algebras. There are two types of representations. For the first type, the conformal dimensions $\Delta$ and $\bar{\Delta}$ are constant along the multicritical line. The second

**Table 1.** Spin, normalised level at $N = 12$, extrapolated value of even $N$ sequence and level identification for the low-lying levels of the MHS model at $\lambda = 7\pi/12$ ($X_{\mathrm{T}} = 8/15$), for the sectors $(R, R') = (1, 1)$, $(1, -1)$ and $(-1, 1) = (-1, -1)$.

| | | | | Level identification | |
|---|---|---|---|---|---|
| $(R, R')$ | $S$ | $X_r\ (12)$ | Extrapolated | Representation | Exact |
| $(1, 1)$ | 0 | 0.5348 | 0.533 | $X_{2,0}$ | $0.5\dot{3}$ |
| | 0 | 1.964 | — | $X_{0,1}$ | 1.875 |
| | 0 | 2.012 | 2.01 | $(1, 1)$ | 2 |
| | 0 | 2.254 | 2.15 | $X_{4,0}$ | $2.1\dot{3}$ |
| | 0 | 2.750 | 2.62 | $X_{2,0}+1+1$ | $2.5\dot{3}$ |
| | 1 | 1.627 | 1.56 | $X_{2,0}+1+0$ | $1.5\dot{3}$ |
| | 1 | 3.094 | 3.02 | $X_{0,1}+1+0$ | 2.875 |
| | 2 | 2.165 | 2.05 | $(0+2, 0)$ | 2 |
| | 2 | 2.579 | 2.54 | $X_{2,1}$ | $2.408\dot{3}$ |
| | 2 | 2.687 | 2.54 | $X_{2,0}+2+0$ | $2.5\dot{3}$ |
| $(1, -1)$ | 0 | 0.1321 | 0.133 | $X_{1,0}$ | $0.1\dot{3}$ |
| | 0 | 1.225 | 1.20 | $X_{3,0}$ | 1.2 |
| | 0 | 2.191 | 2.14 | $X_{1,0}+1+1$ | $2.1\dot{3}$ |
| | 1 | 1.150 | 1.14 | $X_{1,0}+1+0$ | $1.1\dot{3}$ |
| | 1 | 2.140 | 2.04 | $X_{1,1}$ | $2.008\dot{3}$ |
| | 1 | 2.447 | 2.30 | $X_{3,0}+1+0$ | 2.2 |
| | 2 | 2.148 | — | $X_{1,0}+2+0$ | $2.1\dot{3}$ |
| | 2 | 2.467 | 2.32 | $X_{1,0}+2+0$ | $2.1\dot{3}$ |
| $(-1, 1)$ | 0 | 0.1244 | 0.125 | $(\frac{1}{16}, \frac{1}{16})$ | 0.125 |
| $=(-1, -1)$ | 0 | 1.138 | 1.125 | $(\frac{9}{16}, \frac{9}{16})$ | 1.125 |
| | 0 | 2.242 | 2.14 | $(\frac{1}{16}+1, \frac{1}{16}+1)$ | 2.125 |
| | 1 | 1.175 | 1.13 | $(\frac{1}{16}+1, \frac{1}{16})$ | 1.125 |
| | 1 | 2.163 | 2.13 | $(\frac{9}{16}+1, \frac{9}{16})$ | 2.125 |
| | 1 | 2.326 | 2.20 | $(\frac{25}{16}, \frac{9}{16})$ | 2.125 |
| | 2 | 2.303 | 2.187 | $(\frac{1}{16}+2, \frac{1}{16})$ | 2.125 |
| | 2 | 2.641 | — | $(\frac{1}{16}+2, \frac{1}{16})$ | 2.125 |

is of the form $(\Delta, \bar{\Delta}) = (h_{n,m}, h_{n,-m})$ where

$$h_{n,m} = \frac{1}{4}\left[\left(\frac{X_{\mathrm{T}}}{2}\right)^{1/2} n + \left(\frac{2}{X_{\mathrm{T}}}\right)^{1/2} m\right]^2. \tag{8}$$

For the latter, the scaling dimension is $X_{n,m} = X_{\mathrm{T}} n^2/4 + m^2/X_{\mathrm{T}}$ and the spin is $S_{n,m} = nm$. To each $(\Delta, \bar{\Delta})$, there corresponds a set of levels $\Delta + \bar{\Delta} + r + \bar{r}$ ($r, \bar{r} = 0, 1, 2, \ldots$) whose degeneracy is determined by the character formula (Baake *et al* 1987, Yang 1987). In the fourth columns of table 1, we list the conjectured identification of the levels in terms of the irreducible representations. We show $(\Delta, \bar{\Delta})$ and $X_{n,m}$ for the primary operators of the first and second type, respectively. The descendants are denoted as $(\Delta + r, \bar{\Delta} + \bar{r})$ and $X_{n,m} + r + \bar{r}$, respectively. The last columns of table 1 are the exact scaling dimensions of the levels identified. For other values of $\lambda$, we find the same identifications with similar accuracies. Baake *et al* (1987) have found 13 sectors for the AT model. From table 1, we find that the $(R, R') = (1, 1)$, $(1, -1)$ and $(-1, \pm 1)$ sectors of the MHS model correspond to the sectors $\mathscr{A} + \mathscr{F}$, $\mathscr{C} + \mathscr{G}$ and $\mathscr{H}$, respectively, of Baake *et al.* Our result exactly matches that of the AT model with the periodic boundary condition. The anisotropy operator discussed in table 1($a$) of I should be related to the $S=2$, $X=2$ level in the (1, 1) sector and hence is not primary.

Next, we consider the odd $N$ sequences. Table 2 shows our identifications of several levels in the sectors (1, 1), (1, −1) (−1, 1) and (−1, −1) for $\lambda=7\pi/12$ and $N$ odd. Notations are the same as in table 1. Here, half-integer spins appear in the $R'=-1$ sectors since $N$ is odd. For extrapolation, we use the $N=7, 9, 11$ sequences. From table 2 we identify the four sectors (1, 1), (1,−1), (−1, 1) and (−1, −1) as the sectors $\mathscr{H}$, $\mathscr{K}$, $\mathscr{E}$ and $\mathscr{J}$, respectively, of Baake *et al.* These constitute the operator content of the AT model with the boundary condition $\sigma_{N+1}=-\sigma_1$, $s_{N+1}=s_1$ where $\sigma$ and $s$ are the Ising spin variables of the AT model (Yang 1987, Baake *et al* 1987).

In summary, we find numerically that the transfer matrix spectra of the multicritical MHS model are exactly the same as those of the multicritical AT model. Thus apart from the leading non-universal bulk free energy contributions the partition functions on a finite torus in the continuum limit are exactly the same for the MHS and the AT models as long as $X_T$ is the same in the two models. Along the multicritical lines, $X_T$

**Table 2.** Spin, normalised level at $N=11$, extrapolated value of odd $-N$ sequence and level identification for the low-lying levels of the MHS model at $\lambda=7\pi/12$ ($X_T=8/15$), for the sectors $(R, R')=(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. The daggered entry is an exact doublet.

| | | | | Level identification | |
|---|---|---|---|---|---|
| $(R, R')$ | $S$ | $X_r(11)$ | Extrapolated | Representation | Exact |
| (1, 1) | 0 | 0.1251 | 0.125 | $(\frac{1}{16}, \frac{1}{16})$ | 0.125 |
| | 0 | 1.144 | 1.127 | $(\frac{9}{16}, \frac{9}{16})$ | 1.125 |
| | 0 | 2.292 | 2.18 | $(\frac{1}{16}+1, \frac{1}{16}+1)$ | 2.125 |
| | 1 | 1.186 | 1.14 | $(\frac{1}{16}+1, \frac{1}{16})$ | 1.125 |
| | 1 | 2.169 | 2.13 } | { $(\frac{9}{16}+1, \frac{9}{16})$ | 2.125 |
| | 1 | 2.389 | 2.27 } | { $(\frac{25}{16}, \frac{9}{16})$ | 2.125 |
| | 2 | 2.344 | 2.23 } | { $(\frac{1}{16}+2, \frac{1}{16})$ | 2.125 |
| | 2 | 2.800 | — } | { $(\frac{1}{16}+2, \frac{1}{16})$ | 2.125 |
| (1, −1) | $\frac{1}{2}$ | 0.6305 | 0.625 | $(\frac{9}{16}, \frac{1}{16})$ | 0.625 |
| | $\frac{1}{2}$ | 1.703 | 1.64 | $(\frac{1}{16}+1, \frac{9}{16})$ | 1.625 |
| | $\frac{1}{2}$ | 2.755 | 2.67 } | { $(\frac{9}{16}+1, \frac{1}{16}+1)$ | 2.625 |
| | $\frac{1}{2}$ | 2.940 | 2.77 } | { $(\frac{25}{16}, \frac{1}{16}+1)$ | 2.625 |
| | $\frac{3}{2}$ | 1.645 | 1.63 } | { $(\frac{9}{16}+1, \frac{1}{16})$ | 1.625 |
| | $\frac{3}{2}$ | 1.849 | 1.74 } | { $(\frac{25}{16}, \frac{1}{16})$ | 1.625 |
| (−1, 1) | 0 | 0.46996 | 0.4688 | $X_{0,1/2}$ | 0.468 75 |
| | 0 | 2.074† | 2.02 | { $X_{2,1/2}+0+1$ | $2.002\,08\dot{3}$ |
| | | | | { $X_{2,-1/2}+1+0$ | |
| | 0 | 2.698 | 2.54 | $X_{0,1/2}+1+1$ | 2.468 75 |
| | 1 | 1.051 | 1.01 | $X_{2,1/2}$ | $1.002\,08\dot{3}$ |
| | 1 | 1.566 | 1.50 | $X_{0,1/2}+1+0$ | 1.468 75 |
| | 2 | 2.512 | 2.47 | $X_{2,1/2}+1+0$ | $2.002\,08\dot{3}$ |
| (−1, −1) | $\frac{1}{2}$ | 0.6125 | 0.603 | $X_{1,1/2}$ | $0.602\,08\dot{3}$ |
| | $\frac{1}{2}$ | 1.642 | 1.61 | $X_{1,-1/2}+1+0$ | $1.602\,08\dot{3}$ |
| | $\frac{1}{2}$ | 2.899 | 2.77 | $X_{1,1/2}+1+1$ | $2.602\,08\dot{3}$ |
| | $\frac{1}{2}$ | 2.930 | 2.81 | $X_{3,1/2}+0+1$ | 2.668 75 |
| | $\frac{3}{2}$ | 1.820 | 1.73 | $X_{1,1/2}+1+0$ | $1.602\,08\dot{3}$ |
| | $\frac{3}{2}$ | 1.860 | 1.74 | $X_{3,1/2}$ | 1.668 75 |

varies continuously in the range $\frac{2}{9} \leqslant X_T \leqslant \frac{2}{3}$ for the MHS model as given by (7) compared with $\frac{1}{2} \leqslant X_T \leqslant \frac{3}{2}$ for the AT model on a lattice. It is interesting to note that the special point $M$ discussed in Pearce (1985, 1987) and in I is the supersymmetric point with $X_T = \frac{1}{3}$ (Yang and Zheng 1987, Baake *et al* 1987).

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